WNE Linear Algebra
Final Exam

28 January 2020

Questions A

Question 1.
Is the matrix $A = \begin{bmatrix} a^2 & a \\ a & 2 \end{bmatrix}$ positive definite for all $a \in \mathbb{R}$?

Answer 1.
No, it is not. For $a = 0$ the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix},$$

is not positive definite.

Question 2.
If $v, w \in, \mathbb{R}^3$, $v \neq \mathbf{0}$, is the image of vector $w \in \mathbb{R}^3$ under the (linear) orthogonal symmetry about the line $V = \text{lin}(v)$ equal to

$$S_V(w) = 2\frac{w \cdot v}{v \cdot v}v - w?$$

Answer 2.
Yes, it is. An orthogonal basis of $V = \text{lin}(v)$ is the single vector $v \in V$, therefore

$$P_V(w) = \frac{w \cdot v}{v \cdot v}v,$$

and (see Lecture 10)

$$S_V(w) = 2P_V(w) - w,$$

which gives

$$S_V(w) = 2\frac{w \cdot v}{v \cdot v}v - w.$$

Question 3.
If $A \in M(2 \times 2; \mathbb{R})$ and $A + A^{\intercal} = \mathbf{0}$, does it follow that $A^3 + (A^{\intercal})^3 = \mathbf{0}$?

Answer 3.
Yes, it does. If $A^{\intercal} = -A$ then

$$A^3 + (A^{\intercal})^3 = A^3 + (-A)^3 = A^3 - A^3 = \mathbf{0}.$$

Question 4.
Is it possible that $A, B \in M(2 \times 2; \mathbb{R})$, $\det A \neq 0$, $\det B \neq 0$ but $\det(A + B) = 0$?

Answer 4.
Yes, it is.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Question 5.
Does there exist matrix $A \in (100 \times 3; \mathbb{R})$ with pairwise different rows such that the dimension of the set of all solutions of the equation $Ax = \mathbf{0}$ is equal to 1?

Answer 5.
Yes, it does.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \\ \vdots & \ddots & \vdots \\ 99 & 99 & 99 \end{bmatrix}.$$

Question 6.
Are the affine subspaces $E,\ H \subset \mathbb{R}^3$ given by

$$E\colon x_1 + 2x_2 - 3x_3 = 2,$$
$$H = (1,2,1) + \operatorname{lin}((1,2,-3)),$$

perpendicular?

Answer 6.
Yes, they are. The subspace E has a parametrization

$$(-2x_2 + 3x_3 + 2, x_2, x_3) = (2,0,0) + x_2(-2,1,0) + x_3(3,0,1), \quad x_2, x_3 \in \mathbb{R},$$

that is

$$E = (2,0,0) + \operatorname{lin}(((-2,1,0),(3,0,1)).$$

The directions

$$\vec{E} = \operatorname{lin}(((-2,1,0),(3,0,1)),$$
$$\vec{H} = \operatorname{lin}((1,2,-3)),$$

are perpendicular because

$$(1,2,-3)\cdot(-2,1,0) = (1,2,-3)\cdot(3,0,1) = 0.$$

Questions B

Question 1.
Is the matrix $A = \begin{bmatrix} -2 & a \\ a & -a^2 \end{bmatrix}$ negative definite for all $a \in \mathbb{R}$?

Answer 1.
No, it is not. For $a = 0$ the matrix

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix},$$

is not negative definite.

Question 2.
If $v, w \in, \mathbb{R}^3,\ \|v\| = 1$, is the image of vector $w \in \mathbb{R}^3$ under the (linear) orthogonal projection on the line $V = \operatorname{lin}(v)$ equal to

$$P_V(w) = (w \cdot v)v?$$

Answer 2.
Yes, it is. An orthonormal basis of $V = \operatorname{lin}(v)$ is the single vector $v \in V$, therefore

$$P_V(w) = \frac{w \cdot v}{v \cdot v} v,$$

and

$$1 = \|v\|^2 = v \cdot v,$$

which gives

$$P_V(w) = (w \cdot v)v.$$

Question 3.
If $A \in M(2 \times 2; \mathbb{R})$ and $A + A^\intercal = \mathbf{0}$, does it follow that $A^2 + (A^\intercal)^2 = \mathbf{0}$?

Answer 3.
No, it does not. If $A = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$ then $A + A^\intercal = \mathbf{0}$, but

$$A^2 + (A^\intercal)^2 = A^2 + (-A)^2 = 2A^2 = \begin{bmatrix} 0 & -a^2 \\ -a^2 & 0 \end{bmatrix},$$

which is in general not equal to $\mathbf{0}$.

Question 4.
Is it possible that $A, B \in M(2 \times 2; \mathbb{R})$, $\det A = 0$, $\det B = 0$ but $\det(A + B) \neq 0$?

Answer 4.
Yes, it is.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad A + B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Question 5.
Does there exist matrix $A \in (100 \times 3; \mathbb{R})$ with pairwise different rows such that the dimension of the set of all solutions of the equation $Ax = \mathbf{0}$ is equal to 2?

Answer 5.
Yes, it does.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ \vdots & \ddots & \vdots \\ 99 & 99 & 99 \\ 100 & 100 & 100 \end{bmatrix}.$$

Question 6.
Are the affine subspaces $E,\ H \subset \mathbb{R}^3$ given by

$$E\colon x_1 + x_2 - 2x_3 = 5,$$
$$H = (1, -1, 0) + \operatorname{lin}((1, 1, 1)),$$

perpendicular?

Answer 6.
No, they are not. The subspace E has a parametrization

$$(-x_2 + 2x_3 + 5, x_2, x_3) = (5, 0, 0) + x_2(-1, 1, 0) + x_3(2, 0, 1), \quad x_2, x_3 \in \mathbb{R},$$

that is

$$E = (5, 0, 0) + \operatorname{lin}(((-1, 1, 0), (2, 0, 1)).$$

The directions

$$\vec{E} = \operatorname{lin}(((-1, 1, 0), (2, 0, 1)),$$
$$\vec{H} = \operatorname{lin}((1, 1, 1)),$$

are not perpendicular because, for example

$$(1, 1, 1) \cdot (2, 0, 1) = 3 \neq 0.$$